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A piecewise polynomial approach to analyzing interpolatory subdivision

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Abstract

The four-point interpolatory subdivision scheme of Dubuc and its generalizations to irregularly spaced data studied by Warren and by Daubechies, Guskov, and Sweldens are based on fitting cubic polynomials locally. In this paper, we analyze the convergence of the scheme by viewing the limit function as the limit of piecewise cubic functions arising from the scheme. This allows us to recover the regularity results of Daubechies et al. in a simpler way and to obtain the approximation order of the scheme and its first derivative.

Keywords: Interpolatory subdivision; Piecewise polynomials

1. Introduction

The subdivision scheme studied by Daubechies et al. [1] is a generalization of the four-point scheme of Dubuc [3] to irregularly spaced grids of points. Suppose we are given the values $f(x_k)$, $k \in \mathbb{Z}$, of some real function $f : \mathbb{R} \to \mathbb{R}$ at an increasing sequence of grid points

$$\cdots < x_{-1} < x_0 < x_1 < \cdots .$$

For convenience, we assume that f has compact support, so that only a finite number of the values $f(x_k)$ are non-zero. We initialize the subdivision scheme by setting $x_{0,k} = x_k$ and $g_{0,k} = f(x_k)$

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for $k \in \mathbb{Z}$. Then, for each subdivision level $j \geq 0$, we choose new grid points from the old ones by the rules $x_{j+1,2k} = x_{j,k}$ and

$$x_{j,k} < x_{j+1,2k+1} < x_{j,k+1},$$

and compute new values $g_{j+1,k}$ from the old values $g_{j,k}$ using cubic polynomial interpolation. We let $g_{j+1,2k} = g_{j,k}$ and let $g_{j+1,2k+1}$ be the value at the point $x_{j+1,2k+1}$ of the unique cubic polynomial that has the value $g_{j,i}$ at the point $x_{j,i}$ for $i = k-1, k, k+1, k+2$.

The central question about this scheme, as with many others, is that of convergence. A subdivision scheme is said to converge if it has a continuous limit function, which, in this case, is a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $g(x_{j,k}) = g_{j,k}$ for all $j \geq 0$ and $k \in \mathbb{Z}$. In particular, since $g(x_k) = f(x_k)$, this particular scheme is interpolatory. The scheme has been studied in several papers [3,2,5,10,11,1,9,12] and it has been shown that it converges under various conditions on the grid

$$X := \{x_{j,k} : j \geq 0, k \in \mathbb{Z}\}.$$

It has also been shown, again under restrictions on the grid, that the limit function g is continuously differentiable.

Specifically, Dubuc introduced the scheme on the *regular* grid, i.e., the grid in which $x_{j,k} = 2^{-j}k$, and showed that g has Hölder regularity $C^{2-\epsilon}$ for any small $\epsilon > 0$. Dyn et al. [5] also studied the regular scheme as a special case of a family of schemes that include a tension parameter. They showed that g is C^1 but not in general twice differentiable. Later Warren [11] considered the scheme on a *semi-regular* grid, in which the points $x_{0,k}$ are arbitrary but $x_{j+1,2k+1} = (x_{j,k} + x_{j,k+1})/2$ for all $j \geq 0$ and $k \in \mathbb{Z}$, and argued that the C^1 continuity of g continues to hold in this case.

Daubechies et al. [1] introduced the idea of a *dyadically balanced* grid. If $h_{j,k} = x_{j,k+1} - x_{j,k}$ and

$$\lambda = \sup_{j,k} \max\left(\frac{h_{j+1,2k}}{h_{j,k}}, \frac{h_{j+1,2k+1}}{h_{j,k}}\right),$$

then $1/2 \leq \lambda \leq 1$, and the grid is dyadically balanced if $\lambda < 1$ (the quantity $\beta = 1 - \lambda$ was used in the definition in [1]). It was shown in [1] that g is again C^1 if the grid is dyadically balanced. It was further shown that if $\lambda \leq 2/3$ then the derivative g' is Hölder continuous with exponent $1 - \epsilon$ for any small $\epsilon > 0$. This recovers the result of Dubuc because $\lambda = 1/2$ for a regular (and semi-regular) grid.

The purpose of this paper is to offer a new way of establishing these convergence results. Instead of viewing g as the limit of polygons, we treat it as the limit of piecewise cubic functions, built from the cubic polynomials used to define the scheme. This approach appears at first to complicate the analysis. However, it has the advantage that differences between successive piecewise cubics and their derivatives can be expressed in terms of the fourth order divided differences of the scheme. Thus, we can, and do, use the bounds derived in [1] on the growth rate of these differences, but we avoid the need for the 'reduction strategy' used in [1] to convert these bounds to bounds on the growth or decay of lower order differences, and with it the need for 'homogeneity'. Moreover, we derive the scheme for fourth order differences directly from simple properties of the interpolating cubic polynomials. *We never use the Lagrange form of the basic scheme*.

Using this piecewise cubic approach, we also derive a new result for non-regular grids: namely the *approximation order* of the scheme and its first derivative; see Theorem 2. We also make an

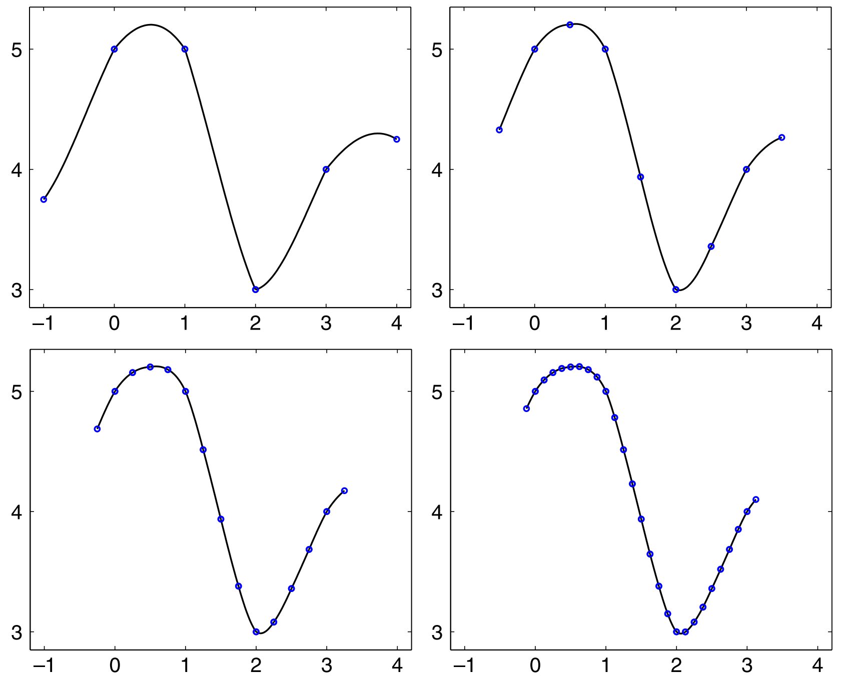

Fig. 1. The sequence of piecewise cubics.

improvement on the $\lambda \le 2/3$ condition of Daubechies et al. We show that g has regularity $C^{2-\epsilon}$ even if $\lambda \le \lambda_0 \approx 0.7142$.

2. A piecewise cubic approach

Let $p_{j,k}^{[m]}$ denote the polynomial of degree $\le m$ that interpolates the value $g_{j,i}$ at the point $x_{j,i}$ for $i = k, k+1, \ldots, k+m$. Then the subdivision scheme can be expressed as

$$g_{j+1,2k} = g_{j,k},$$
$$g_{j+1,2k+1} = p_{j,k-1}^{[3]}(x_{j+1,2k+1}).$$

Let $I_{j,k} = [x_{j,k}, x_{j,k+1}]$ and let $s_j : \mathbb{R} \to \mathbb{R}$ denote the piecewise cubic function defined by

$$s_j(x) = p_{j,k-1}^{[3]}(x), \quad x \in I_{j,k}.$$

The first few s_j of an example data set are shown in Fig. 1. We will show that the sequence of piecewise cubics s_j converges to a continuous limit function g as $j \to \infty$ under some assumptions on the spacing of the grid points. Thus we want to show that the functions s_j form a Cauchy sequence in the max norm. This motivates finding a useful expression for the differences $s_{j+1} - s_j$. To this end, we define the nodal polynomials

$$\psi_{j,k}^{[m]}(x) := (x - x_{j,k})(x - x_{j,k+1}) \cdots (x - x_{j,k+m}),$$

the differences

$$h_{j,k}^{[m]} := x_{j,k+m} - x_{j,k},$$

and the divided differences

$$g_{j,k}^{[m]} = \frac{g_{j,k+1}^{[m-1]} - g_{j,k}^{[m-1]}}{h_{j,k}^{[m]}}, \quad m \geq 1,$$

and $g_{j,k}^{[0]} = g_{j,k}$. We will sometimes also need to consider divided differences over non-consecutive points. For any distinct integers $i_0, i_1, \ldots, i_m$, let $[i_0, i_1, \ldots, i_m]g_{j,k}$ denote the divided difference of the values $g_{j,k+i_0}, \ldots, g_{j,k+i_m}$ at the corresponding points $x_{j,k+i_0}, \ldots, x_{j,k+i_m}$. So $[i]g_{j,k} = g_{j,k+i}$ and for $m \geq 1$,

$$[i_0, i_1, \ldots, i_m]g_{j,k} = \frac{[i_1, i_2, \ldots, i_m]g_{j,k} - [i_0, i_1, \ldots, i_{m-1}]g_{j,k}}{x_{j,k+i_m} - x_{j,k+i_0}}.$$

In particular, $[0, 1, \ldots, m]g_{j,k} = g_{j,k}^{[m]}$.

Lemma 1. *For $j \geq 0$,*

$$s_{j+1}(x) - s_j(x) = \begin{cases} \psi_{j+1,2k}^{[2]}(x) h_{j+1,2k-2} g_{j+1,2k-2}^{[4]}, & x \in I_{j+1,2k}; \\ -\psi_{j+1,2k}^{[2]}(x) h_{j+1,2k+3} g_{j+1,2k}^{[4]}, & x \in I_{j+1,2k+1}. \end{cases} \tag{1}$$

Proof. Let $x \in I_{j+1,2k}$. Then

$$s_{j+1}(x) - s_j(x) = p_{j+1,2k-1}^{[3]}(x) - p_{j,k-1}^{[3]}(x),$$

and since $p_{j+1,2k-1}^{[3]}$ and $p_{j,k-1}^{[3]}$ agree at the points $x_{j,k}, x_{j+1,2k+1}$, and $x_{j,k+1}$,

$$s_{j+1}(x) - s_j(x) = c\psi_{j+1,2k}^{[2]}(x),$$

for some constant c. Moreover, c must be the leading coefficient of the polynomial $p_{j+1,2k-1}^{[3]} - p_{j,k-1}^{[3]}$, and so

$$c = g_{j+1,2k-1}^{[3]} - g_{j,k-1}^{[3]} = [1, 2, 3, 4]g_{j+1,2k-2} - [0, 2, 4, 6]g_{j+1,2k-2}.$$

Now, by the definition of $g_{j+1,2k+1}$, we have

$$[0, 2, 4, 6]g_{j+1,2k-2} = [0, 2, 3, 4]g_{j+1,2k-2},$$

from which it follows that

$$c = (x_{j+1,2k-1} - x_{j+1,2k-2})[0, 1, 2, 3, 4]g_{j+1,2k-2} = h_{j+1,2k-2} g_{j+1,2k-2}^{[4]}.$$

The case that $x \in I_{j+1,2k+1}$ is similar. $\square$

From this result we see that successive differences between the piecewise cubics s_j can be expressed in terms of the fourth order divided differences of the scheme. The convergence analysis therefore reduces to the question of the rate of growth of these differences, which was also an important ingredient of the analysis in [1]. To analyze the differentiability of g and the regularity of the derivative g', we will also study the behavior of the first and second derivatives of s_j, which are piecewise quadratic and piecewise linear respectively, and are in general discontinuous at the points $x_{j,k}$. The first few piecewise quadratics s_j' are shown in Fig. 2.

We will see that the s_j' converge, under certain assumptions on the grid points, and that the s_j'' diverge. To obtain the convergence of the s_j' and to bound the growth rate of the s_j'' we can

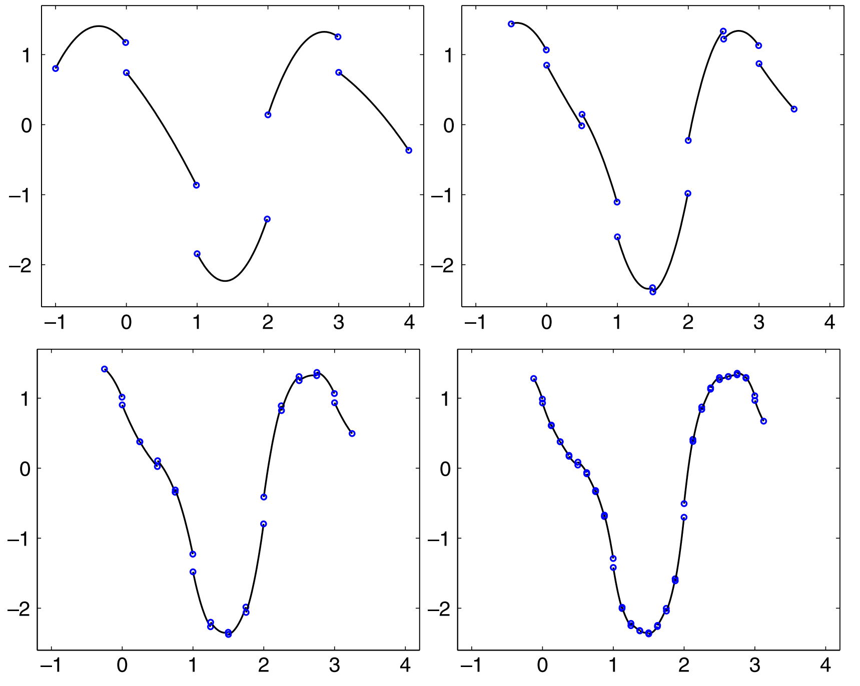

Fig. 2. The first derivative of the first few piecewise cubics.

simply differentiate the Eq. (1) with respect to x, giving expressions for $s'_{j+1} - s'_j$ and $s''_{j+1} - s''_j$, which are also in terms of the fourth order differences of the scheme.

Thus, a further ingredient in the analysis is a bound on the cubic $\psi^{[2]}_{j+1,2k}$ in Lemma 1 and its derivatives.

Lemma 2. *For* $j \geq 0$ *and* $x \in I_{j+1,2k}$,

$$|(\psi^{[2]}_{j+1,2k})^{(r)}(x)| \leq A_r h^{2-r}_{j+1,2k} h_{j,k}, \quad r = 0, 1, 2,$$

where $A_0 = 1$, $A_1 = 3$, *and* $A_2 = 6$.

Proof. These inequalities follow easily from differentiating the formula

$$\psi^{[2]}_{j+1,2k}(x) = (x - x_{j+1,2k})(x - x_{j+1,2k+1})(x - x_{j+1,2k+2}). \quad \square$$

When studying specifically the behavior of the s'_j we must also control the sizes of the jumps in s'_j at the break points $x_{j,k}$. This motivates us to derive an expression for these jumps. We will denote by $s'_{j,-}(x)$ and $s'_{j,+}(x)$ the left and right derivatives of s_j at $x \in \mathbb{R}$ respectively. Since s_j is a cubic polynomial in each interval $I_{j,k}$, it has both a right and a left derivative at every $x \in \mathbb{R}$. For $x \in (x_{j,k}, x_{j,k+1})$, these two derivatives are equal.

We will now, and later, work with the differences of divided differences,

$$\tilde{g}^{[m]}_{j,k} := g^{[m-1]}_{j,k+1} - g^{[m-1]}_{j,k} = h^{[m]}_{j,k} g^{[m]}_{j,k}$$

(which were also used extensively in [1]).

Lemma 3. *For* $j \geq 0$,

$$s'_{j,+}(x_{j,k}) - s'_{j,-}(x_{j,k}) = -h_{j,k-1}h_{j,k}\tilde{g}^{[4]}_{j,k-2}. \tag{2}$$

Proof. By the Newton form, we can express $s_j(x)$ for $x \in I_{j,k}$ as

$$s_j(x) = p^{[2]}_{j,k-1}(x) + \psi^{[2]}_{j,k-1}(x)g^{[3]}_{j,k-1},$$

and differentiating this at $x = x_{j,k}$ gives

$$s'_{j,+}(x_{j,k}) = (p^{[2]}_{j,k-1})'(x_{j,k}) + (\psi^{[2]}_{j,k-1})'(x_{j,k})g^{[3]}_{j,k-1}. \tag{3}$$

For x in $I_{j,k-1}$, we can express s_j as

$$s_j(x) = p^{[2]}_{j,k-1}(x) + \psi^{[2]}_{j,k-1}(x)g^{[3]}_{j,k-2},$$

and differentiating this at $x = x_{j,k}$ implies

$$s'_{j,-}(x_{j,k}) = (p^{[2]}_{j,k-1})'(x_{j,k}) + (\psi^{[2]}_{j,k-1})'(x_{j,k})g^{[3]}_{j,k-2}. \tag{4}$$

We then obtain (2) by subtracting (4) from (3) and using the fact that

$$(\psi^{[2]}_{j,k-1})'(x_{j,k}) = -h_{j,k-1}h_{j,k}. \quad \square$$

Thus we see that the jumps in the first derivative of s_j at the points $x_{k,j}$ can also be expressed in terms of fourth order differences.

In view of Lemmas 1 and 3, we need to bound fourth order divided differences. As shown in [1], there are subdivision schemes for divided differences of all orders up to and including order 4. These schemes were derived in [1] by starting with the Lagrange form of the initial scheme and recursively applying symbolic manipulation to obtain the first, second, third, and fourth order schemes. In this paper we only need the fourth order scheme and we give an independent and direct derivation of it using similar ideas to the proof of Lemma 1.

Lemma 4. *For* $j \geq 0$,

$$\tilde{g}^{[4]}_{j+1,2k} = \frac{h^{[4]}_{j+1,2k}}{h^{[2]}_{j+1,2k+1}}\tilde{g}^{[4]}_{j,k-1}, \tag{5}$$

$$\tilde{g}^{[4]}_{j+1,2k+1} = -\frac{h_{j+1,2k}}{h^{[2]}_{j+1,2k+1}}\tilde{g}^{[4]}_{j,k-1} - \frac{h_{j+1,2k+5}}{h^{[2]}_{j+1,2k+3}}\tilde{g}^{[4]}_{j,k}. \tag{6}$$

Proof. The even case, Eq. (5), follows from the fact that

$$\begin{aligned} g^{[4]}_{j+1,2k} &= \frac{[0,2,3,4]g_{j+1,2k} - [0,1,2,4]g_{j+1,2k}}{h^{[2]}_{j+1,2k+1}} \\ &= \frac{g^{[3]}_{j,k} - g^{[3]}_{j,k-1}}{h^{[2]}_{j+1,2k+1}} = \frac{\tilde{g}^{[4]}_{j,k-1}}{h^{[2]}_{j+1,2k+1}}. \end{aligned}$$

The odd case (6) follows from

$$\begin{aligned}\tilde{g}^{[4]}_{j+1,2k+1} &= g^{[3]}_{j+1,2k+2} - g^{[3]}_{j+1,2k+1} \\ &= -(g^{[3]}_{j+1,2k+1} - g^{[3]}_{j,k}) - (g^{[3]}_{j,k} - g^{[3]}_{j+1,2k+2}) \\ &= -(g^{[3]}_{j+1,2k+1} - [0,2,3,4]g_{j+1,2k}) - ([0,1,2,4]g_{j+1,2k+2} - g^{[3]}_{j+1,2k+2}) \\ &= -h_{j+1,2k}g^{[4]}_{j+1,2k} - h_{j+1,2k+5}g^{[4]}_{j+1,2k+2},\end{aligned}$$

and an application of (5). □

3. Convergence criteria

In this section we derive some general criteria for when the piecewise cubics s_j converge to a continuous limit function g, and when g is C^1. We will use the norm $\|s\| := \sup_{x\in\mathbb{R}} |s(x)|$ for a bounded function $s : \mathbb{R} \to \mathbb{R}$.

Lemma 5. *Suppose there are constants $C_0 > 0$ and $\beta, 0 < \beta < 1$, such that*

$$\|s_{j+1} - s_j\| \le C_0\beta^j, \quad j \ge 0. \tag{7}$$

Then there is a continuous limit function

$$g(x) := \lim_{j\to\infty} s_j(x), \quad x \in \mathbb{R}, \tag{8}$$

and the rate of convergence is $O(\beta^j)$ as $j \to \infty$; specifically,

$$\|g - s_j\| \le C_0\beta^j/(1-\beta). \tag{9}$$

Proof. The bound (7) implies that the sequence of continuous functions $s_0, s_1, s_2, \ldots$ is uniformly Cauchy because it implies that for any $m < n$,

$$\|s_n - s_m\| \le \sum_{j=m}^{n-1} \|s_{j+1} - s_j\| \le C_0\beta^m/(1-\beta), \tag{10}$$

which can be made arbitrarily small by taking m large enough. From this follows the existence of the continuous limit function g. Since the bound (10) holds for any $n > m$, it also holds in the limiting case that s_n is replaced by g, which establishes (9). □

To give a criterion for the differentiability of g, we use the decay of both the differences $s'_{j+1} - s'_j$ and the jumps in s'_j at the points $x_{j,k}$.

Lemma 6. *Suppose, in addition to the hypothesis of Lemma 5, that there are constants $C_1 > 0$ and $\gamma, 0 < \gamma < 1$, such that*

$$\|s'_{j+1,\pm} - s'_{j,\pm}\| \le C_1\gamma^j, \quad j \ge 0, \tag{11}$$

and suppose further that for all grid points $x \in X$,

$$|s'_{j,+}(x) - s'_{j,-}(x)| \to 0 \quad \text{as } j \to \infty. \tag{12}$$

Then the limit function g in (8) is continuously differentiable and

$$\|g' - s'_{j,\pm}\| \le C_1\gamma^j/(1-\gamma). \tag{13}$$

Proof. Due to (11), even though the functions $s'_{j,+}$ and $s'_{j,-}$ are not continuous, they form uniformly Cauchy sequences, by a similar reasoning to that of Lemma 5. They therefore have pointwise limits

$$\phi_{\pm}(x) := \lim_{j\to\infty} s'_{j,\pm}(x), \quad x \in \mathbb{R},\ j \geq 0,$$

with the property that

$$\|\phi_{\pm} - s'_{j,\pm}\| \leq C_1\gamma^j/(1-\gamma), \quad j \geq 0. \tag{14}$$

We next show that $\phi_+ = \phi_-$, using (12) to controls the jumps in $(s_j)'$. Suppose first that $x \notin X$. Then $s'_{j,+}(x) = s'_{j,-}(x)$ for all $j \geq 0$ and so $\phi_+(x) = \phi_-(x)$. Otherwise $x = x_{J,K}$ for some $J \geq 0$ and $K \in \mathbb{Z}$. Then for any $j \geq J$, $x = x_{j,k}$ where $k = 2^{j-J}K$, and

$$|\phi_+(x) - \phi_-(x)| \leq |\phi_+(x) - s'_{j,+}(x)| + |s'_{j,+}(x) - s'_{j,-}(x)| + |s'_{j,-}(x) - \phi_-(x)|.$$

Due to (12) and (14), the right hand side can be made arbitrarily small by choosing j large enough and so $\phi_+(x) = \phi_-(x)$ in this case too.

We can now define $\phi := \phi_+ = \phi_-$, and we next show that ϕ is continuous. Let $x \in \mathbb{R}$ and let $\epsilon > 0$. By (14), there is some $j \geq 0$ such that

$$\|\phi - s'_{j,+}\| \leq \epsilon/3.$$

Then, since $s'_{j,+}$ is continuous to the right of x, there is some $\delta_+ > 0$ such that if $y \in (x, x+\delta_+)$,

$$|s'_{j,+}(y) - s'_{j,+}(x)| \leq \epsilon/3.$$

Hence

$$|\phi(y) - \phi(x)| \leq |\phi(y) - s'_{j,+}(y)| + |s'_{j,+}(y) - s'_{j,+}(x)| + |s'_{j,+}(x) - \phi_+(x)| \leq \epsilon.$$

Similarly, working with $s'_{j,-}$ instead of $s'_{j,+}$, there is also some $\delta_- > 0$ such that if $y \in (x-\delta_-, x)$ then $|\phi(y) - \phi(x)| \leq \epsilon$. This shows that ϕ is continuous.

It remains to show that g is differentiable with $g' = \phi$, which we can do by showing that

$$g(x) - g(x_{0,0}) = \int_{x_{0,0}}^{x} \phi(y)\,\mathrm{d}y, \quad x \in \mathbb{R}. \tag{15}$$

Since both sides of Eq. (15) are continuous in x, it is sufficient to show that it holds for all grid points $x = x_{J,K}$. We assume that $x > x_{0,0}$, the other case being similar, so that $K > 0$. Then for any $j \geq J$, we have $x = x_{j,k}$, where $k = 2^{J-j}K > 0$, and so

$$g(x) - g(x_{0,0}) = \sum_{i=0}^{k-1}(g_{j,i+1} - g_{j,i}) = \sum_{i=0}^{k-1}(x_{j,i+1} - x_{j,i})s'_j(\xi_{j,i}),$$

for some $\xi_{j,i} \in (x_{j,i}, x_{j,i+1})$, and so

$$g(x) - g(x_{0,0}) = A + B,$$

where

$$A = \sum_{i=0}^{k-1}(x_{j,i+1} - x_{j,i})\phi(\xi_{j,i}),$$

and

$$B = \sum_{i=0}^{k-1} (x_{j,i+1} - x_{j,i})(s_j'(\xi_{j,i}) - \phi(\xi_{j,i})).$$

Now, as $j \to \infty$, since ϕ is a continuous function, A converges to the integral in (15) and

$$|B| \le \|s_{j,+}' - \phi\| \sum_{i=0}^{k-1} (x_{j,i+1} - x_{j,i}) = \|s_{j,+}' - \phi\|(x - x_{0,0}) \to 0. \quad \square$$

4. Convergence for dyadically balanced grids

In this section we reproduce the convergence result of [1] for dyadically balanced grids. We start with a lemma that was essentially proved in Lemma 6 of [1]. We give an independent proof here because we need to be specific about the constant C involved, in order to prove later the approximation result, Theorem 2. Let

$$h = \sup_k h_{0,k} \quad \text{and} \quad M := \sup_k |\tilde{g}_{0,k}^{[4]}|.$$

Lemma 7. *Suppose $\lambda < 1$. Then for all $j \ge 0$ and $k \in \mathbb{Z}$,*

$$|\tilde{g}_{j,k}^{[4]}| \le \frac{C\lambda^j}{h_{j,k+1} h_{j,k+2}},$$

where $C = h^2 M$.

Proof. Let

$$G_{j,k} := h_{j,k+1} h_{j,k+2} \tilde{g}_{j,k}^{[4]}.$$

Then from (5)–(6), we obtain a scheme for $G_{j,k}$. For fixed j and k,

$$G_{j+1,2k} = aG_{j,k-1},$$
$$G_{j+1,2k+1} = -bG_{j,k-1} - cG_{j,k},$$

where

$$a = \frac{h_{j+1,2k+1} h_{j+1,2k+2} h_{j+1,2k}^{[4]}}{h_{j+1,2k+1}^{[2]} h_{j,k} h_{j,k+1}},$$

and

$$b = \frac{h_{j+1,2k+2} h_{j+1,2k+3} h_{j+1,2k}}{h_{j+1,2k+1}^{[2]} h_{j,k} h_{j,k+1}}, \qquad c = \frac{h_{j+1,2k+2} h_{j+1,2k+3} h_{j+1,2k+5}}{h_{j+1,2k+3}^{[2]} h_{j,k+1} h_{j,k+2}}.$$

Considering a, since

$$\frac{h_{j+1,2k}^{[4]}}{h_{j+1,2k+1}^{[2]}} = \frac{h_{j,k} + h_{j,k+1}}{h_{j+1,2k+1} + h_{j+1,2k+2}} \le \max\left(\frac{h_{j,k}}{h_{j+1,2k+1}}, \frac{h_{j,k+1}}{h_{j+1,2k+2}}\right),$$

it follows that

$$a \le \max\left(\frac{h_{j+1,2k+2}}{h_{j,k+1}}, \frac{h_{j+1,2k+1}}{h_{j,k}}\right) \le \lambda.$$

Further,

$$b \le \frac{h_{j+1,2k+3}h_{j+1,2k}}{h_{j,k}h_{j,k+1}} \le \frac{h_{j+1,2k+3}}{h_{j,k+1}}\lambda,$$

and

$$c \le \frac{h_{j+1,2k+2}h_{j+1,2k+5}}{h_{j,k+1}h_{j,k+2}} \le \frac{h_{j+1,2k+2}}{h_{j,k+1}}\lambda,$$

and therefore, $b + c \le \lambda$. Hence,

$$\sup_k |G_{j+1,k}| \le \lambda \sup_k |G_{j,k}| \le \lambda^{j+1} \sup_k |G_{0,k}| \le \lambda^{j+1}h^2M. \quad \square$$

This lemma and the results of the previous sections now give us the following.

Theorem 1. *If $\lambda < 1$, the scheme has a C^1 limit function g and, moreover,*

$$\|g - s_j\| \le h^3 M\lambda^{2j+2}/(1 - \lambda^2), \tag{16}$$

and

$$\|g' - s'_{j,\pm}\| \le 3h^2 M\lambda^{j+1}/(1 - \lambda). \tag{17}$$

Proof. In order to apply Lemmas 5 and 6, let $x \in (x_{j+1,2k}, x_{j+1,2k+1})$ and consider the first case of (1). From Eq. (5) we have

$$|g^{[4]}_{j+1,2k-2}| = \frac{|\tilde{g}^{[4]}_{j,k-2}|}{h^{[2]}_{j+1,2k-1}} \le \frac{|\tilde{g}^{[4]}_{j,k-2}|}{h_{j+1,2k}}. \tag{18}$$

This and Lemma 2 then show that

$$|s^{(r)}_{j+1}(x) - s^{(r)}_j(x)| \le A_r h_{j+1,2k-2}h_{j,k}h^{1-r}_{j+1,2k}|\tilde{g}^{[4]}_{j,k-2}|, \quad r = 0, 1, 2. \tag{19}$$

Therefore, since $h_{j+1,2k-2} \le \lambda h_{j,k-1}$, Lemma 7 implies

$$|s^{(r)}_{j+1}(x) - s^{(r)}_j(x)| \le A_r h^{1-r}_{j+1,2k}h^2M\lambda^{j+1}, \quad r = 0, 1, 2. \tag{20}$$

Now, to apply Lemma 5, we let $r = 0$ and noting that $h_{j+1,2k} \le \lambda^{j+1}h$, we have

$$|s_{j+1}(x) - s_j(x)| \le h^3M\lambda^{2(j+1)}.$$

The same inequality holds for $x \in I_{j+1,2k+1}$ and so (7) holds with $\beta = \lambda^2$ and $C_0 = \lambda^2h^3M$, and therefore the scheme has a continuous limit function g satisfying (16).

Next we want to apply Lemma 6. The case $r = 1$ of (20) gives

$$|s'_{j+1}(x) - s'_j(x)| \le 3h^2M\lambda^{j+1},$$

and so (11) holds with $\gamma = \lambda$ and $C_1 = 3\lambda h^2M$. Further, (12) holds because by Lemmas 3 and 7,

$$|s'_{j,+}(x_{j,k}) - s'_{j,-}(x_{j,k})| \le h^2M\lambda^j. \tag{21}$$

Thus the criteria for Lemma 6 are fulfilled and g is C^1 and satisfies (17). $\square$

5. Approximation order

With the machinery developed so far it is now quite easy to derive the approximation order of the scheme in the dyadically balanced case, the idea being to compare both f and g with s_0.

Theorem 2. *If $\lambda < 1$ and f has a bounded fourth derivative in $\mathbb{R}$, there are constants $C_0, C_1 > 0$, that depend only on λ, such that*

$$\|f - g\| \le C_0 h^4 \|f^{(4)}\|, \tag{22}$$

$$\|f' - g'\| \le C_1 h^3 \|f^{(4)}\|. \tag{23}$$

For regular grids the estimate (22) was established in [5] using a quasi-interpolant approach.

Proof. We let $x \in (x_{0,k}, x_{0,k+1})$ and use the triangle inequality,

$$|f^{(r)}(x) - g^{(r)}(x)| \le |f^{(r)}(x) - s_0^{(r)}(x)| + |g^{(r)}(x) - s_0^{(r)}(x)|, \quad r = 0, 1. \tag{24}$$

Starting with the case $r = 0$, consider the first term on the right. The Newton error formula gives

$$f(x) - s_0(x) = \psi_{0,k-1}^{[3]}(x) f^{(4)}(\xi)/4!,$$

for some $\xi \in (x_{0,k-1}, x_{0,k+2})$. The inequalities

$$|x - x_{0,k}|\,|x - x_{0,k+1}| \le h^2/4 \quad \text{and} \quad |x - x_{0,k-1}|\,|x - x_{0,k+2}| \le 9h^2/4,$$

then lead to the estimate,

$$|\psi_{0,k-1}^{[3]}(x)| \le 9h^4/16,$$

and consequently,

$$|f(x) - s_0(x)| \le 9h^4 \|f^{(4)}\|/384. \tag{25}$$

Consider the second term on the right of (24). Eq. (16) in the case $j = 0$ is

$$\|g - s_0\| \le h^3 M \lambda^2/(1 - \lambda^2),$$

and since

$$M = \sup_k (h_{0,k}^{[4]} |g_{0,k}^{[4]}|) \le 4h \sup_k |g_{0,k}^{[4]}| \le h\|f^{(4)}\|/6, \tag{26}$$

it follows that

$$\|g - s_0\| \le h^4 \|f^{(4)}\| \lambda^2/(6(1 - \lambda^2)).$$

Combining this with (25) gives (22) with

$$C_0 = \frac{9}{384} + \frac{\lambda^2}{6(1 - \lambda^2)}.$$

Now let $r = 1$ in (24). To bound the first term on the right we use an inequality of Isaacson and Keller [7, Thm. 1, Sec. 6.5], which shows that

$$f'(x) - s_0'(x) = \prod_{i=0}^{2} (x - \eta_i) f^{(4)}(\xi)/3!,$$

for some $\xi \in (x_{0,k-1}, x_{0,k+2})$ and $x_{0,i} < \eta_i < x_{0,i+1}$, $i = k-1, k, k+1$. Since

$$|x - \eta_0|\,|x - \eta_2| \le 9h^2/4 \quad \text{and} \quad |x - \eta_1| \le h,$$

this leads to

$$|f'(x) - s_0'(x)| \le 3h^3 \|f^{(4)}\|/8. \tag{27}$$

Considering the second term on the right of (24), Eq. (17) in the case $j = 0$ gives

$$|g'(x) - s_0'(x)| \le 3\lambda h^2 M/(1-\lambda),$$

and using (26) this leads to

$$|g'(x) - s_0'(x)| \le \lambda h^3 \|f^{(4)}\|/(2(1-\lambda)).$$

Combining this with (27) gives (23) with

$$C_1 = \frac{3}{8} + \frac{\lambda}{2(1-\lambda)}. \quad \square$$

6. Hölder regularity

In this section we recover the Hölder regularity of the dyadically balanced scheme derived in [1]. In this and the next section, C and D will denote constants that are independent of j and k. We also use the notation $k_j(x) = \max\{\ell : x_{j,\ell} \le x\}$ for $x \in \mathbb{R}$.

Lemma 8. *If $\lambda < 1$, then for $j \ge 1$ and $x \in (x_{j,k}, x_{j,k+1})$,*

$$|s_j''(x)| \le Cj \frac{\lambda^{j-1}}{h_{j,k}} + D. \tag{28}$$

Proof. First we show that for $j \ge 1$ and $x \in (x_{j,k}, x_{j,k+1})$,

$$|s_j''(x) - s_{j-1}''(x)| \le C \frac{\lambda^{j-1}}{h_{j,k}}. \tag{29}$$

To see this, we see that for $j \ge 0$ and $x \in (x_{j+1,2k}, x_{j+1,2k+1})$, the case $r = 2$ of (20) gives

$$|s_{j+1}''(x) - s_j''(x)| \le C \frac{\lambda^j}{h_{j+1,2k}}, \tag{30}$$

which is (29) in the case that k is even. From the second case of (1), a similar analysis shows that (29) also holds when k is odd.

Now observe that for $x \in (x_{j,k}, x_{j,k+1})$, applying (29) repeatedly gives

$$|s_j''(x) - s_0''(x)| \le \sum_{i=1}^{j} |s_i''(x) - s_{i-1}''(x)| \le C \sum_{i=1}^{j} \frac{\lambda^{i-1}}{h_{i,k_i(x)}},$$

and since

$$\frac{1}{h_{i,k_i(x)}} \le \frac{\lambda}{h_{i+1,k_{i+1}(x)}} \le \cdots \le \frac{\lambda^{j-i}}{h_{j,k}},$$

this means that

$$|s_j''(x) - s_0''(x)| \le Cj \frac{\lambda^{j-1}}{h_{j,k}}$$

which implies (28). □

We can now derive the first Hölder result of [1].

Theorem 3. *If $\lambda < 1$, the function g' is Hölder continuous with exponent $\alpha = \log\lambda / \log(1-\lambda) - \epsilon$ for any small $\epsilon > 0$.*

Proof. Let $x, y \in \mathbb{R}$ be such that $x < y$ and $y - x \le h_\star$ where

$$h_\star := \inf_k h_{0,k}.$$

For each $j \ge 0$, let $n_j(x, y)$ denote the number of points $x_{j,k}$ that belong to the open interval (x, y). Since $y - x \le h_\star$ we have $n_0(x, y) \in \{0, 1\}$. Further, $n_j(x, y) \le n_{j+1}(x, y) \le 2n_j(x, y) + 1$ and $n_j(x, y) \to \infty$ as $j \to \infty$, and therefore there must be some $j \ge 1$ such that $n_j(x, y) \in \{2, 3\}$. Then, letting $r = n_j(x, y)$, there is some $k \in \mathbb{Z}$ such that

$$x_{j,k} \le x < x_{j,k+1} < \cdots < x_{j,k+r} < y \le x_{j,k+r+1}.$$

By the triangle inequality,

$$|g'(y) - g'(x)| \le |g'(y) - s'_{j,-}(y)| + |s'_{j,-}(y) - s'_{j,+}(x)| + |s'_{j,+}(x) - g'(x)|, \tag{31}$$

and we estimate the middle term. Let $y_0 = x$, $y_i = x_{j,k+i}$, $i = 1, \ldots, r$, and $y_{r+1} = y$. Then

$$\begin{aligned} s'_{j,-}(y) - s'_{j,+}(x) &= \sum_{i=0}^{r} (s'_{j,-}(y_{i+1}) - s'_{j,+}(y_i)) + \sum_{i=1}^{r} (s'_{j,+}(y_i) - s'_{j,-}(y_i)) \\ &= \sum_{i=0}^{r} (y_{i+1} - y_i) s_j''(\xi_i) + \sum_{i=1}^{r} (s'_{j,+}(y_i) - s'_{j,-}(y_i)), \end{aligned} \tag{32}$$

for some $\xi_i \in (y_i, y_{i+1})$. By Lemma 8 applied to $(x_{j,k+i}, x_{j,k+i+1})$,

$$(y_{i+1} - y_i)|s_j''(\xi_i)| \le h_{j,k+i} |s_j''(\xi_i)| \le Cj\lambda^{j-1} + h_{j,k+i} D \le (C'j + D')\lambda^j$$

and by (21),

$$|s'_{j,+}(y_i) - s'_{j,-}(y_i)| \le C\lambda^j,$$

and so, since $r \le 3$,

$$|s'_{j,-}(y) - s'_{j,+}(x)| \le (Cj + D)\lambda^j,$$

and by (17) we have

$$|g'(y) - g'(x)| \le (Cj + D)\lambda^j.$$

Therefore, since

$$y - x \ge h_{j,k+1} \ge (1-\lambda)^j h_\star,$$

we have

$$\frac{|g'(y) - g'(x)|}{(y-x)^\alpha} \le (Cj + D) \left(\frac{\lambda}{(1-\lambda)^\alpha} \right)^j.$$

This gives the result because the right hand side is bounded as a function of $j \geq 0$ if $\lambda/(1-\lambda)^\alpha < 1$, or equivalently $\alpha < \log\lambda/\log(1-\lambda)$. $\square$

7. Improved Hölder regularity

Theorem 3 shows that in the regular and semi-regular cases, when $\lambda = 1/2$, the limit function g has regularity $C^{2-\epsilon}$, but for larger values of λ it shows a weaker regularity. In this section we show that g is $C^{2-\epsilon}$ for any $\lambda \leq \lambda_0 \approx 0.7142$. This is equivalent to the condition that $\beta \geq \beta_0 \approx 0.2858$ using the notation β of [1], which improves a little on the condition $\beta \geq 1/3$, required in [1]. We start with a lemma that is similar to Lemma 8 of [1] but that does not require the homogeneity condition of [1].

Lemma 9. *Suppose $\lambda \leq \lambda_0 \approx 0.7142$. Then for all $j \geq 0$ and $k \in \mathbb{Z}$,*

$$|\tilde{g}^{[4]}_{j,k}| \leq \frac{C}{h^{[2]}_{j,k+1}},$$

where $C = hM$.

Proof. Let

$$G_{j,k} := h^{[2]}_{j,k+1}\tilde{g}^{[4]}_{j,k}.$$

From (5)–(6) we obtain a scheme for $G_{j,k}$. For fixed j and k,

$$G_{j+1,2k} = G_{j,k-1},$$
$$G_{j+1,2k+1} = -aG_{j,k-1} - bG_{j,k},$$

where

$$a = \frac{h_{j+1,2k}h^{[2]}_{j+1,2k+2}}{h^{[2]}_{j+1,2k+1}h^{[2]}_{j,k}}, \qquad b = \frac{h^{[2]}_{j+1,2k+2}h_{j+1,2k+5}}{h^{[2]}_{j+1,2k+3}h^{[2]}_{j,k+1}},$$

and the task is to show that $a + b \leq 1$. With $\mu = h_{j+1,2k+2}/h_{j,k+1}$,

$$a \leq \frac{\lambda h_{j,k}h_{j,k+1}}{((1-\lambda)h_{j,k} + \mu h_{j,k+1})h^{[2]}_{j,k}} = \frac{\lambda}{(1-\lambda)}F(R),$$

where

$$F(R) = \frac{R}{(R+S)(R+1)},$$

and $R = h_{j,k}/h_{j,k+1}$ and $S = \mu/(1-\lambda)$. For $R > 0$, the function F achieves its maximum when $R = \sqrt{S}$, and so $F(R) \leq F(\sqrt{S})$, and therefore

$$a \leq \frac{\lambda}{(1-\lambda)(1+\sqrt{S})^2} = \frac{\lambda}{(\sqrt{1-\lambda}+\sqrt{\mu})^2}.$$

Similarly,

$$b \leq \frac{\lambda}{(\sqrt{1-\lambda}+\sqrt{1-\mu})^2},$$

and therefore,

$$a+b \le \max_{1-\lambda \le \mu \le \lambda} G(\mu),$$

where

$$G(\mu) = \frac{\lambda}{(\sqrt{1-\lambda}+\sqrt{\mu})^2} + \frac{\lambda}{(\sqrt{1-\lambda}+\sqrt{1-\mu})^2}.$$

Since the second derivative of G is non-negative, G is convex and so

$$a+b \le \max(G(1-\lambda), G(\lambda)) = G(\lambda) = \gamma\left(\frac{1}{(1+\sqrt{\gamma})^2} + \frac{1}{4}\right),$$

where $\gamma = \lambda/(1-\lambda)$. As observed in [1], the right hand side is increasing in γ and is ≤ 1 for $\gamma \le \gamma_0 \approx 2.4992$. This condition is equivalent to the condition that $\lambda \le \lambda_0 = \gamma_0/(1+\gamma_0) \approx 0.7142$. □

The goal is to show that the derivative g' is Hölder continuous with exponent $1-\epsilon$ under the assumption that λ is in the range $\lambda \le \lambda_0$. First we need some preliminary results.

Lemma 10. *If* $\lambda \le \lambda_0$ *then*

$$|s'_{j+1}(x) - s'_j(x)| \le C h_{j,k}, \quad x_{j,k} < x < x_{j,k+1}, \tag{33}$$

and

$$|s'_{j+1}(x) - s'_j(x)| \le C h_{j+1,2k-2}, \quad x_{j+1,2k} < x < x_{j+1,2k+1}. \tag{34}$$

Proof. Both follow immediately from inequality (19) and Lemma 9. □

Lemma 11. *If* $\lambda \le \lambda_0$ *then*

$$|g'(x) - s'_j(x)| \le C h_{j,k}, \quad x_{j,k} < x < x_{j,k+1}. \tag{35}$$

Proof. Due to (33), for $n > j$,

$$|s'_n(x) - s'_j(x)| \le \sum_{i=j}^{n-1} |s'_{i+1}(x) - s'_i(x)| \le C \sum_{i=j}^{n-1} h_{i,k_i(x)} \le C h_{j,k} \sum_{i=j}^{n-1} \lambda^{i-j},$$

which gives (35) by letting $n \to \infty$. □

Lemma 12. *If* $\lambda \le \lambda_0$ *then*

$$|g'(x) - s'_j(x)| \le C(x - x_{j,k-1}), \quad x_{j,k} < x < x_{j,k+1}. \tag{36}$$

Proof. There is some $n \ge j$ such that $k_i(x) = 2k_{i-1}(x)$ for $i = j+1, \ldots, n$, and $k_{n+1}(x) = 2k_n(x) + 1$. Then

$$|g'(x) - s'_j(x)| \le |g'(x) - s'_n(x)| + |s'_n(x) - s'_j(x)|,$$

and by Lemma 11,

$$|g'(x) - s'_n(x)| \le C h_{n,2^{n-j}k} \le C h_{n+1,2^{n+1-j}k}/(1-\lambda) \le C(x - x_{j,k})/(1-\lambda),$$

and by Lemma 10,

$$|s_n'(x) - s_j'(x)| \le \sum_{i=j}^{n-1} |s_{i+1}'(x) - s_i'(x)| \le C \sum_{i=j}^{n-1} h_{i+1,2^{i+1-j}k-2} \le C h_{j,k-1},$$

which gives (36). □

Theorem 4. *If $\lambda \le \lambda_0$, the function g' is Hölder continuous with exponent $1-\epsilon$ for any small $\epsilon > 0$.*

Proof. We return to the triangle inequality (31) and the expression for the middle term, (32). From Lemma 9, using (2), we have for $j \ge 0$,

$$|s_{j,+}'(x_{j,k}) - s_{j,-}'(x_{j,k})| \le C \frac{h_{j,k-1} h_{j,k}}{h_{j,k-1}^{[2]}} \le C \min\{h_{j,k-1}, h_{j,k}\}, \tag{37}$$

and it follows that

$$|s_{j,+}'(y_i) - s_{j,-}'(y_i)| \le C(y-x)$$

in (32). Further, applying Lemma 9 to the case $r = 2$ of (19) gives

$$|s_{j+1}''(x) - s_j''(x)| \le C,$$

and therefore

$$|s_j''(x)| \le Cj + D.$$

Applying these two estimates to (32), shows that

$$|s_{j,-}'(y) - s_{j,+}'(x)| \le (Cj + D)(y - x).$$

Next, observe that Lemma 12 shows that

$$|g'(y) - s_{j,-}'(y)| \le C(y - x_{j,k+r-1}) \le C(y-x)$$

in (31), and a similar argument shows that

$$|s_{j,+}'(x) - g'(x)| \le C(y-x).$$

Then

$$|g'(y) - g'(x)| \le (Cj + D)(y - x).$$

Since

$$y - x \le x_{j,k+r+1} - x_{j,k} \le (r+1)\lambda^j h,$$

this means that

$$\frac{|g'(y) - g'(x)|}{(y-x)^\alpha} \le (Cj + D)\lambda^{j(1-\alpha)},$$

the right hand side of which is bounded as a function of $j \ge 0$ for any $\alpha \in (0,1)$. □

8. Final remarks

Any interpolatory subdivision scheme that is based on a local interpolation method could be analyzed using the same basic approach as here: by building functions piecewise, interval by interval, from the local interpolants defining the scheme, and studying their asymptotic behavior. Whether or not this turns out to be beneficial is a topic for future research. Examples of schemes that fall into this category are: the family of schemes of Deslauriers and Dubuc [2] which are based on polynomial interpolation of odd degree; the convexity-preserving scheme of [6] (see also [8,13]) which is based on rational interpolation, with a quadratic numerator and linear denominator; and the nonlinear curve scheme in [4] which is based on parametric cubic interpolation.

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